# A basic consideration on the quaternion algebra

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## 1 Basic algebra in 4 dimensions

A quaternion algebra can be described as a 4-dimensional vector space with a canonical base {1, *i*, *j*, *k* } having the following Hamilton's multiplication rules [R3]:

$i^2 = -1$ $j^2 = -1$ $ij = k$ $ji = -k$ (non commutativness)

Quaternion Multiplication table

| x.y= | 1 | i | j | k |
|---|---|---|---|---|
| 1 | 1 | i | j | k |
| i | i | -1 | k | -j |
| j | j | -k | -1 | i |
| k | k | j | -i | -1 |

The base components *i*, *j*, *k* can be seen as the complex numbers; similarly, a concept of conjugate can be defined.

- Conjugate: with $Q = q_0 + q_1 i + q_2 j + q_3 k$ the conjugate is defined by $\overline{Q} = q_0 - q_1 i - q_2 j - q_3 k$.
- Opposite: it is defined by $-Q = -q_0 - q_1 i - q_2 j - q_3 k$
- Imaginary quaternion *:* when $q_0$=0.
- Quaternion product (basic form): it is a distributive non-commutative product using explicitly the Hamilton's rules 1,*i*, *j*, *k* with the multiplication table.

For example $Q.\overline{Q} = (q_0 + q_1 i + q_2 j + q_3 k).(q_0 - q_1 i - q_2 j - q_3 k) = q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2$

Other example: $\overline{Q_1.Q_2} = \overline{Q}_2.\overline{Q}_1$

There are no ambiguities with the definitions (or axioms) on a 4-dimensional canonical base, this is useful for fundamental algebra purpose, however this is quite heavy to use and there is no obvious meaning. Moreover, for some other users, the base is different { *i, j, k,* 1 } leading to confusions $\mathrm{Q}_{\text{other}} = q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k} + q_4$ . They may also use $Q_{other} = \{\vec{q}, q_4\}$

## 2 Second representation with a real and a vector

One can write a quaternion as $Q$ = {$q_0$, $\vec{q}$ } where $q_0$ is the real part, and $\vec{q}$ is a vector having the components {$q_1$, $q_2$, $q_3$} in the imaginary canonical base *i*, *j*, *k* . This base can however be regarded as equivalent to any geometrical base, for example the base of a 3-dimensional Cartesian frame. Hence a link between quaternion and a real geometric world is more obvious.

Note*: an imaginary quaternion* $\mathrm{q}$ *is written as* $\mathrm{q} = \{0, \vec{q}\} = ½\left(Q - \overline{Q}\right)$ *where* $\vec{q}$ *is the unique vector corresponding to the imaginary quaternion* $\mathrm{q}$.

- Quaternion product second form:

$Q_1.Q_2$={$q_0$, $\vec{q}$ }.{ $w_0$, $\vec{w}$ }= {$q_0.w_0$ - $\vec{q}\cdot\vec{w}$ , $q_0$ $\vec{w}$ + $w_0$ $\vec{q}$ + $\vec{q}\times\vec{w}$ }

where " $\cdot$ " stand for the scalar vector dot product and " $\times$" the vector cross product (right handed as usual).

Note: *the product of two imaginary quaternion* $\{0, \vec{q}\}.\{0, \vec{w}\}$ *is simply :*$\{-\vec{q}\cdot\vec{w}, \vec{q}\times\vec{w}\}$*.*

*To get only the imaginary part of this quaternion (i.e. the cross product only), one shall consider the quaternion*: $\{0, \vec{q}\times\vec{w}\} = ½\left(Q_1.Q_2 - \overline{Q_1.Q_2}\right)$ *i.e.* $\{0, \vec{q}\times\vec{w}\} = ½\left(q.w - \overline{q.w}\right)$.

- Quaternion product matrix:
  *It can be performed with standard matrix product "* * *"* $Q_1.Q_2 = [Q_1] * Q_2 = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$
  *where* $[Q_1]$ *is a skew symmetric matrix defined with the coordinates of* $Q_1$ *(a bit similar to the multiplication table) and with* $Q_2$ *written as a column matrix.*

  *Also with writing* $Q_1.Q_2$*=*$\{w_0.q_0 - \vec{w}\cdot\vec{q}\ ,\ w_0\vec{q} + q_0\vec{w} - \vec{w}\times\vec{q}\}$
  *one can define with* $Q_2$ *a matrix* $[\hat{Q}_2]$ *:* $[\hat{Q}_2] * Q_1 = \begin{bmatrix} w_0 & -w_1 & -w_2 & -w_3 \\ w_1 & w_0 & w_3 & -w_2 \\ w_2 & -w_3 & w_0 & w_1 \\ w_3 & w_2 & -w_1 & w_0 \end{bmatrix} * \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$

  *The two matrixes are not identical because* $[Q_1]$ *is based on the coordinates of* $Q_1$ and $[\hat{Q}_2]$ *is based on the coordinates of* $Q_2$. *Moreover, one shall remember that the product is not commutative in general.*
- Unit quaternion: Among the quaternion, the ones used here are the unit quaternion: the norm being defined as $\|Q\| = Q.\overline{Q}$, a unit quaternion has a norm=1.
- Quaternion inverse: $Q.Q^{-1} = Q^{-1}.Q = 1 \qquad Q^{-1} = \overline{Q}/\|Q\|^2$ ; for unit quaternion $Q^{-1} = \overline{Q}$ .
- Vectors and coordinates: Considering a first inertial frame "*i*" and a second frame for a body "*b*" having an instantaneous rotation $\vec{\Omega}$.

  An abstract vector written with the letter $\vec{V}$ is a very concrete concept that does not depends on any frame. But operationally, the vector belongs to a vector space (here a 3-dimensions), so that the frame in which one write its coordinates is of prime importance: every scalar product, cross product, matrix form product, local derivative and even quaternion operation shall be carefully performed within the same vector space i.e. within the same frame used to write the coordinates of the vectors.

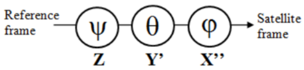
More explicitly let's use as superscript the frame in which the coordinates are written and used. The vector with coordinates written in the inertial frame$^i$ is $\vec{V}^i$ ; $\vec{V}^b$ is the same vector $\vec{V}$ but with coordinates written in frame$^b$ and $\vec{\Omega}^b_{b/i}$ the instantaneous rotation of the frame$^b$ with respect to frame$^i$ but with coordinates written in frame$^b$. It is obvious to say that even if it is the same vector, the coordinates are in general not equal $\vec{V}^i \neq \vec{V}^b$, etc .

## 3 Third representation with an angle θ and a vector

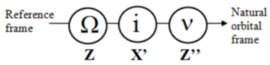
For unit quaternion that are used now on, $Q = \{\cos\theta/2, \sin\theta/2\,\vec{u}\}$ where $\vec{u}$ is a unit vector. This can be written shortly with the following $Q(\theta, \vec{u})$. With this form, the quaternion gives the orientation of a body with respect to a first frame$^i$: that is a rotation of angle $\theta$ around $\vec{u}$ considered as a vector with coordinates written in the first frame $\vec{u} = \vec{u}^i$ (because this axis is invariant by $Q$, either $\vec{u} = \vec{u}^b = \vec{u}^i$ with the same coordinates in the two frames).

*Note: there is not a unique unit quaternion giving an orientation: other possibility is to consider the rotation of angle 2π- θ around the axis* $-\vec{u}$ *: that is: a quaternion or its opposite* *give the same orientation. Thus, when it is useful to keep the unit quaternion uniqueness for giving orientation: a possible rule is to select the quaternion having a positive first component* $\cos\theta/2$ *>= 0 and if not to use the opposite one.*

To orient: The meaning of a quaternion product $Q_1.Q_2$ is that it gives the orientation of a third frame$^b$ wrt a first frame$^i$, (*body* wrt *inertial*) performed with $Q_1(\theta, \vec{u})$ and then with $Q_2(\phi, \vec{w})$ where $\vec{u} = \vec{u}^i$ unit vector axis in frame$^i$ and $\vec{w} = \vec{w}^o$ is a unit vector axis in the second frame$^o$ (*orbit*). This makes obvious the link between Euler angles or Cardan angles and the quaternion. The advantage of the unit quaternion is that because they always have a norm of 1, the special cases of indetermination (gimbals lock) with the other transformations can't occur anymore. Thus for a whole quaternion $Q_{i,b} = Q_{i,o}.\ Q_{o,b}$ we can deduce $Q_{o,b} = Q_{i,o}^{-1}.\ Q_{i,b}$ where $Q_{x,y}$ is the quaternion from "*x*" to "*y*".

Example 1: 

*the successive rotations with the Cardan angles* $\psi,\theta,\varphi$ *around the axes Z (yaw), then Y' (pitch) then X'' (roll) in the Euler's sequence* (3; 2; 1) *give the quaternion* $Q_1(\psi, \vec{e}_Z).Q_2(\theta, \vec{e}_Y).Q_3(\varphi, \vec{e}_X)$.

Example 2: 

*the rotations with Euler's angles in Euler sequence* (3; 1; 3) $\Omega, i, \nu$ *around the axes Z (precession), then X' (nutation) then Z'' (spin) give the quaternion* $Q_1(\Omega, \vec{e}_Z).Q_2(i, \vec{e}_X).Q_3(\nu, \vec{e}_Z)$.

*Note: for orbital purpose, Ω and i are the same, but here ν Is not the true anomaly, it should be understood as the angle ω+ν, with ω the perigee argument.*

Vectors derivation: One knows that the derivative of a vector $\vec{V}$ depends on the frame in which the derivation is performed. $\dot{\vec{V}}^b{}_{/i} = \left.\dfrac{d\vec{V}^b}{dt}\right|_{/i} = \left.\dfrac{d\vec{V}^b}{dt}\right|_{/b} + \vec{\Omega}^b_{b/i} \times \vec{V}^b$ where in $\dfrac{d}{dt}{}_{/x}$ indice $_{/x}$ stand for derivation reference frame, and where $\vec{\Omega}^b_{b/i}$ is the instantaneous rotation of the body frame wrt the inertial frame but with coordinates written in the body frame.

## 4 Sandwich product

The expression of a vector after an orientation given by a quaternion is the **sandwich** product

The rule for orienting a vector $\vec{V}^i$ from "*i*" to "*b*" by a quaternion *Q*= $Q_{i,b}$ is given by the sandwiching product: $V'^i = Q.V^i.\overline{Q}$ where $V^i$ and $V'^i$ (without the vector arrow) are here the corresponding imaginary quaternion to the vectors $\vec{V}^i$ and $\vec{V}'^i$.

A subtle but evident relation is to be mentioned: we consider the frame "*b*" that is the frame "*i*" after the orientation defined by the quaternion $Q_{i,b}$. Because the orientations change from "*i*" to "*b*" affect the whole frame, the coordinates of a vector before the rotation in the frame "*i*" and after rotation but in the frame "*b*" are always the same, we have $V'^b = V^i$. Thus, one also has $V'^i = Q.V'^b.\overline{Q}$. Either this is true for any vector, thus $V^i = Q.V^b.\overline{Q}$. For example $\dot{V}^i{}_{/i} = Q\dot{V}^b{}_{/i}.\overline{Q}$. Inversely, the other form of sandwiching product, $V^b = \overline{Q}.V^i.Q$ can be used to get the rotation matrix: $V^b = [R_{i,b}] * V^i$ with $[R_{i,b}] = [\overline{Q}] * [\hat{Q}]$

## 5 Derivation with respect to the time [R1], [R2]

In the case of a quaternion $Q$ (=$Q_{i,b}$) that give the orientation of the body frame from the inertial frame, $Q$ depend on the time because the body frame is mobile. To represent the orientation change, the quaternion $Q(\theta, \vec{u})$ can be written with $\vec{u} = \vec{u}^{\,i}$ a vector of the inertial frame base (so that the inertial derivatives of those base vectors are null). Either $\left.\frac{\mathrm{d}Q}{\mathrm{dt}}\right|_{/i} = \dot{Q} = \dot{\mathrm{q}}_0 + \dot{\mathrm{q}}_1 i + \dot{\mathrm{q}}_2 j + \dot{\mathrm{q}}_3 k$ *(attention $\dot{Q}$ is a non-unit quaternion)*.

## 6 Derivation of quaternion, relation with instantaneous rotation

From $V^i = Q.V^b.\overline{Q}$ one can say that $\dot{V}^i{}_{/i} = \dot{Q}.V^b.\overline{Q} + Q.V^b.\dot{\overline{Q}}$ because derivative $\dot{V}^b_{/b}$ is null, $V^b$ being fixed in the rigid body "*b*". It follows $\dot{V}^i{}_{/i} = \dot{Q}.V^b.\overline{Q} - \overline{\dot{Q}.V^b.\overline{Q}}$, keeping in mind that $\overline{V^b} = -V^b$ and $\dot{\overline{Q}} = \overline{\dot{Q}}$. And one has also $\dot{\vec{V}}^b{}_{/i} = \vec{\Omega}^b_{b/i} \times \vec{V}^b$ from the vector derivation, as the vector $\vec{V}^b$ is constant (i.e. fixed in the rigid body frame "*b*").

This last cross product can be written in quaternion algebra, as seen before, $\dot{V}^b{}_{/i} = ½\left(\Omega^b_{b/i}.V^b - \overline{\Omega^b_{b/i}.V^b}\right)$ and thus $\dot{V}^i{}_{/i} = ½\left(Q.\Omega^b_{b/i}.V^b.\overline{Q} - \overline{Q.\Omega^b_{b/i}.V^b.\overline{Q}}\right)$. Finally by identification, one set up a remarkable relation in the quaternion algebra: $\dot{Q} = ½\, Q.\Omega^b_{b/i}$ (also $\dot{Q}^{-1} = -½\,\Omega^b_{b/i} \cdot Q^{-1}$) where here $\Omega^b_{b/i}$ represent an imaginary and non-unit quaternion, = $0 + \mathrm{p}i + qj + rk$, that has the same coordinates of the vector $\vec{\Omega}^b_{b/i}$ written in the body frame. Further, $\frac{1}{2} Q.\Omega^b_{b/i} = \frac{1}{2}(\mathrm{q}_0 + \mathrm{q}_1 i + \mathrm{q}_2 j + \mathrm{q}_3 k).(0 + \mathrm{p}i + qj + rk)$, can be performed, as seen before, with matrixes in the form "$Q_1.Q_2 = \left[\hat{Q}_2\right] * Q_1$", thus $\left[\hat{\Omega}^b_{b/i}\right] = \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$ give finally in matrixes: $\dot{Q} = ½\left[\hat{\Omega}^b_{b/i}\right] * Q$. This relation is used in several tools, in particular in [R 3].

## 7 References:


*[R 1] Vernon Chi. "Quaternions and Rotations in 3-Space", 25 September 1998 ; Leandra Vicci, 27 April 2001*

*[R 2] R. Guiziou., "DESS AIR & ESPACE, systèmes de contrôle d'attitude et d'orbite ", Uni. Aix-Marseille III, 2001.*

*[R 3] KopooS, TriaXOrbitaL tool 1989-2021, ESPSS Satellite library*


*La mathématique des quaternions est une discipline étrange... La première fois que vous la découvrez, vous ne comprenez rien... La deuxième fois, vous pensez que vous comprenez, sauf un ou deux points... La troisième fois, vous savez que vous ne comprenez plus rien, mais à ce niveau vous êtes tellement habitué que ça ne vous dérange plus.*
*attribué à Arnold Sommerfeld pour la thermodynamique, vers 1940.*

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